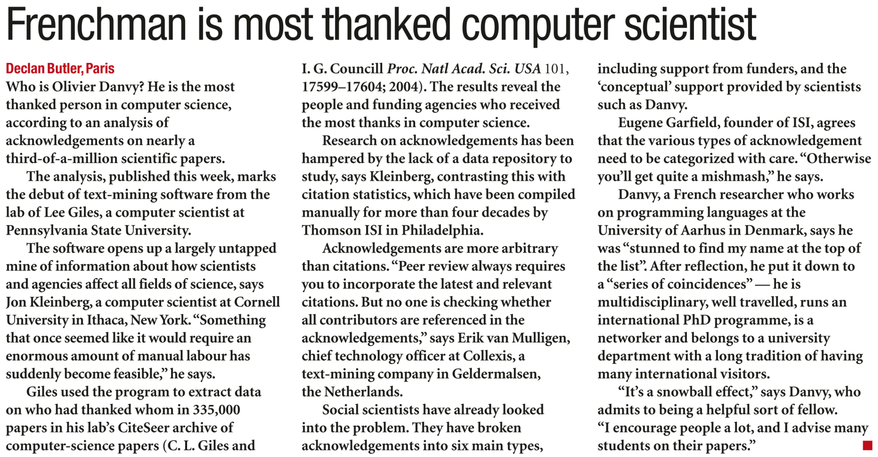
Encoding products

Sam Lindley

Olivierfest, Singapore, October 2025

Prologue

Most thanked computer scientist: Olivier Danvy



Frenchman is most thanked computer scientist

Declan Butler, Paris

Who is Olivier Danvy? He is the most thanked person in computer science, according to an analysis of acknowledgements on nearly a third-of-a-million scientific papers.

The analysis, published this week, marks the debut of text-mining software from the lab of Lee Giles, a computer scientist at Pennsylvania State University.

The software opens up a largely untapped mine of information about how scientists and agencies affect all fields of science, says Jon Kleinberg, a computer scientist at Cornell University in Ithaca, New York. "Something that once seemed like it would require an enormous amount of manual labour has suddenly become feasible," he says.

Giles used the program to extract data on who had thanked whom in 335,000 papers in his lab's CiteSeer archive of computer-science papers (C. L. Giles and I. G. Councill *Proc. Natl Acad. Sci. USA* 101, 17599–17604; 2004). The results reveal the people and funding agencies who received the most thanks in computer science.

Research on acknowledgements has been hampered by the lack of a data repository to study, says Kleinberg, contrasting this with citation statistics, which have been compiled manually for more than four decades by Thomson ISI in Philadelphia.

Acknowledgements are more arbitrary than citations. "Peer review always requires you to incorporate the latest and relevant citations. But no one is checking whether all contributors are referenced in the acknowledgements," says Erik van Mulligen, chief technology officer at Collexis, a text-mining company in Geldermalsen, the Netherlands.

Social scientists have already looked into the problem. They have broken acknowledgements into six main types, including support from funders, and the 'conceptual' support provided by scientists such as Danvy.

Eugene Garfield, founder of ISI, agrees that the various types of acknowledgement need to be categorized with care. "Otherwise you'll get quite a mishmash," he says.

Danvy, a French researcher who works on programming languages at the University of Aarhus in Denmark, says he was "stunned to find my name at the top of the list". After reflection, he put it down to a "series of coincidences" — he is multidisciplinary, well travelled, runs an international PhD programme, is a networker and belongs to a university department with a long tradition of having many international visitors.

"It's a snowball effect," says Danvy, who admits to being a helpful sort of fellow. "I encourage people a lot, and I advise many students on their papers." ■

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- Computability
- Algorithmic complexity
- **Macro expressiveness**
  - **local** versus **global** encodings
  - **compositionality** is a given

  [Felleisen, 1990, "On the expressive power of programming languages"]

Other aspects of expressive power

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Properties

soundness, completeness, contextual equivalence, adequacy, full abstraction, ...

Structure

Galois connections, adjunctions, logical relations, bisimulations, ...

Quiz

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Can simple-typed lambda calculus encode System F?

Can row polymorphism encode row subtyping?

Encoding products

Motivation

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$$\begin{aligned}
\textbf{pair}\ l\ r &\equiv \lambda s.s\ l\ r \\
\textbf{fst}\ p &\equiv p\ (\lambda x.\lambda y.x) \\
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l and r, and hence x and y, must have the same type as the return type of s is fixed:

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Do alternative simply-typed encodings exist for heterogeneous pairs?

Call-by-name CPS

$$\begin{aligned}
\mathcal{N}[\![A]\!] &= (A^\star \to R) \to R \\
X^\star &= X \\
(A \times B)^\star &= \mathcal{N}[\![A]\!] \times \mathcal{N}[\![B]\!] \\
(A \to B)^\star &= \mathcal{N}[\![A]\!] \to \mathcal{N}[\![B]\!]
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$$\begin{array}{rcl}
\mathcal{N}[\![x]\!] & = & \lambda k.x\ k \\
\mathcal{N}[\![\lambda x.M]\!] & = & \lambda k.k\ (\lambda x.\mathcal{N}[\![M]\!]) \\
\mathcal{N}[\![M\ N]\!] & = & \lambda k.\mathcal{N}[\![M]\!]\ (\lambda f.f\ \mathcal{N}[\![N]\!]\ k) \\
\mathcal{N}[\![\mathbf{pair}\ M\ N]\!] & = & \lambda k.k\ (\mathbf{pair}\ \mathcal{N}[\![M]\!]\ \mathcal{N}[\![N]\!]) \\
\mathcal{N}[\![\mathbf{fst}\ M]\!] & = & \lambda k.\mathcal{N}[\![M]\!]\ (\lambda p.(\mathbf{fst}\ p)\ k) \\
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Products are encodable via a curried **global** CPS translation

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\mathcal{C}[\![M\ N]\!] & = & \lambda k.\mathcal{C}[\![M]\!]\ (\lambda f.f\ \mathcal{C}[\![N]\!]\ k) \\
\mathcal{C}[\![\mathbf{pair}\ M\ N]\!] & = & \lambda k.k\ \mathcal{C}[\![M]\!]\ \mathcal{C}[\![N]\!] \\
\mathcal{C}[\![\mathbf{fst}\ M]\!] & = & \lambda k.\mathcal{C}[\![M]\!]\ (\lambda x.\lambda y.x\ k) \\
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\end{array}$$

What about a **local** encoding?

Localising CPS

Untyped

$$
\begin{aligned}
\mathcal{U}[\![\textbf{pair}\ M\ N]\!] &= \lambda s.s\,\mathcal{U}[\![M]\!]\,\mathcal{U}[\![N]\!] \\
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\end{array}$$

Simply typed — homogeneous products

$$\begin{array}{rcl}
\mathcal{H}[\![A \times A]\!] & = & (\mathcal{H}[\![A]\!] \to \mathcal{H}[\![A]\!] \to \mathcal{H}[\![A]\!]) \to \mathcal{H}[\![A]\!] \\
\mathcal{H}[\![\mathbf{pair}\ M^A\ N^A]\!] & = & \lambda s^{\mathcal{H}[\![A]\!] \to \mathcal{H}[\![A]\!] \to \mathcal{H}[\![A]\!]}.s\ \mathcal{H}[\![M]\!]\ \mathcal{H}[\![N]\!] \\
\mathcal{H}[\![\mathbf{fst}\ M^{A \times A}]\!] & = & \mathcal{H}[\![M]\!]\ (\lambda x^{\mathcal{H}[\![A]\!]}.\lambda y^{\mathcal{H}[\![A]\!]}.x) \\
\mathcal{H}[\![\mathbf{snd}\ M^{A \times A}]\!] & = & \mathcal{H}[\![M]\!]\ (\lambda x^{\mathcal{H}[\![A]\!]}.\lambda y^{\mathcal{H}[\![A]\!]}.y)
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Polymorphic

$$\begin{array}{rcl} \mathcal{F}[\![A \times B]\!] & = & \forall Z.(\mathcal{F}[\![A]\!] \to \mathcal{F}[\![B]\!] \to Z) \to Z \\ \mathcal{F}[\![\textbf{pair}_{A,B}\ M\ N]\!] & = & \Lambda Z.\lambda s^{\mathcal{F}[\![A]\!] \to \mathcal{F}[\![B]\!] \to Z}.s\ \mathcal{F}[\![M]\!]\ \mathcal{F}[\![N]\!] \\ \mathcal{F}[\![\textbf{fst}_{A,B}\ M]\!] & = & \mathcal{F}[\![M]\!]\ \mathcal{F}[\![A]\!]\ (\lambda x^{\mathcal{F}[\![A]\!]}.\lambda y^{\mathcal{F}[\![B]\!]}.x) \\ \mathcal{F}[\![\textbf{snd}_{A,B}\ M]\!] & = & \mathcal{F}[\![M]\!]\ \mathcal{F}[\![B]\!]\ (\lambda x^{\mathcal{F}[\![A]\!]}.\lambda y^{\mathcal{F}[\![B]\!]}.y) \end{array}$$

No local encoding of $X \times Y$

We seek β-normal forms $\mathsf{fst}_{X,Y}$ and $\mathsf{snd}_{X,Y}$ such that:

$$\begin{array}{rcl} [\![p : X \times Y \vdash \mathbf{fst}\ p : X]\!] & = & p : [\![X \times Y]\!] \vdash \mathsf{fst}_{X,Y} : X \\ [\![p : X \times Y \vdash \mathbf{snd}\ p : Y]\!] & = & p : [\![X \times Y]\!] \vdash \mathsf{snd}_{X,Y} : Y \end{array}$$

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\end{aligned}$$

So we must have $m, n, M_1, ..., M_m, N_1, ..., N_n$ such that:

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The typing rule for application means that we also have

$$A_1 \to \cdots \to A_m \to X \quad = \quad [\![X \times Y]\!] \quad = \quad B_1 \to \cdots \to B_n \to Y$$

where:

$$\begin{array}{c} (p : [\![X \times Y]\!] \vdash M_i : A_i)_{1 \leq i \leq m} \\ (p : [\![X \times Y]\!] \vdash N_j : B_i)_{1 \leq j \leq n} j \end{array}$$

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But these equations could only hold if X and Y were the same type!

Hang on a minute!

Type-indexed local encodings of products are well-known in PCF and System T.

Examples:

- [Longley and Normann, 2015]
  Higher-order computability
- [Kiselyov, 2021]
  `http://okmij.org/ftp/Computation/simple-encodings.html#product`

How do we reconcile the existence of such encodings with the non-existence result?

No local encoding of $X \times (X \to X)$

Consider $X \times (X \to X)$. We seek β-normal forms $\mathrm{fst}_{X,X\to X}$ and $\mathrm{snd}_{X,X\to X}$ such that:

$$\begin{aligned}
[\![p : X \times (X \to X) \vdash \mathbf{fst}\ p : X]\!] &= p : [\![X \times (X \to X)]\!] \vdash \mathrm{fst}_{X,X\to X} : X \\
[\![p : X \times (X \to X) \vdash \mathbf{snd}\ p : X \to X]\!] &= p : [\![X \times (X \to X)]\!] \vdash \mathrm{snd}_{X,X\to X} : X \to X
\end{aligned}$$

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As before, $\mathsf{fst}_{X,X\to X}$ must be of the form

$$p\ M_1\ \ldots\ M_m$$

and hence:

$$[\![X \times (X \to X)]\!] = A_1 \to \cdots \to A_m \to X$$

No local encoding of $X \times (X \to X)$ (continued)

Two choices for $\mathsf{snd}_{X, X \to X}$:

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No local encoding of $X \times (X \to X)$ (continued)

Two choices for $\mathsf{snd}_{X,X\to X}$:

1. $p\ N_1\ \dots\ N_{m-1}$

$$\begin{aligned} \Longrightarrow\ A_m = X \text{ and } M_m &= p\ M'_1\ \dots\ M'_m \\ M'_m &= p\ M''_1\ \dots\ M''_m \\ &\dots \end{aligned}$$

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No finite such snd can exist.

2. $\lambda z.N'$

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$$\Longrightarrow\ [\![\mathbf{snd}\ (\mathbf{pair}\ x\ y)]\!] = \lambda z.N'[[\![\mathbf{pair}\ x\ y]\!]/p]$$

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   No lambda abstraction can be β-converted to y.

   But what if we allow η-conversion?

Local encoding of $X \times (X \to X)$ with η

$$\mathcal{E}[\![X \times (X \to X)]\!] \quad = \quad (X \to (X \to X) \to X) \to X$$

$$\begin{aligned}
\mathcal{E}[\![\textbf{pair}\ M^{X}\ N^{X \to X}]\!] &= \lambda f.f\ \mathcal{E}[\![M]\!]\ \mathcal{E}[\![N]\!] \\
\mathcal{E}[\![\textbf{fst}\ M^{X \times (X \to X)}]\!] &= \mathcal{E}[\![M]\!]\ (\lambda x\ y.x) \\
\mathcal{E}[\![\textbf{snd}\ M^{X \times (X \to X)}]\!] &= \lambda z.\mathcal{E}[\![M]\!]\ (\lambda x\ y.y\ z)
\end{aligned}$$

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$$
\begin{array}{rcl}
\mathcal{E}[\![X \times (X \to X)]\!] & = & (X \to (X \to X) \to X) \to X \\
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\mathcal{E}[\![\mathbf{fst}\ M^{X \times (X \to X)}]\!] & = & \mathcal{E}[\![M]\!]\ (\lambda x\ y.x) \\
\mathcal{E}[\![\mathbf{snd}\ M^{X \times (X \to X)}]\!] & = & \lambda z.\mathcal{E}[\![M]\!]\ (\lambda x\ y.y\ z)
\end{array}
$$

Now we have

$$
\begin{array}{l}
\mathcal{E}[\![\mathbf{fst}\ (\mathbf{pair}\ x\ y)]\!] \sim_\beta x \\
\mathcal{E}[\![\mathbf{snd}\ (\mathbf{pair}\ x\ y)]\!] \sim_\beta \lambda z.y\ z \sim_\eta y
\end{array}
$$

Local encoding of $A \times B$ with η and a single base type X

$$\begin{array}{rcl}
\mathcal{E}[\![A \times B]\!] & = & (\mathcal{E}[\![A]\!] \to \mathcal{E}[\![B]\!] \to X) \to X \\
\mathcal{E}[\![\textbf{pair}\ M\ N]\!] & = & \lambda f.f\ \mathcal{E}[\![M]\!]\ \mathcal{E}[\![N]\!] \\
\mathcal{E}[\![\textbf{fst}\ M^{A_1 \to \ldots A_m \to X, B}]\!] & = & \lambda z_1\ \ldots\ z_m.\mathcal{E}[\![M]\!]\ (\lambda x\ y.x\ z_1\ \ldots\ z_m) \\
\mathcal{E}[\![\textbf{snd}\ M^{A, B_1 \to \ldots B_n \to X}]\!] & = & \lambda z_1\ \ldots\ z_n.\mathcal{E}[\![M]\!]\ (\lambda x\ y.y\ z_1\ \ldots\ z_n)
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\end{array}$$

This is a **type-indexed** local encoding.

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- Parametric global CPS encoding
- Parametric local Church encodings
  - untyped
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*none of these encodings preserves the η-rule for products

Epilogue

Closing thoughts

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My thoughts on expressiveness are rooted in Olivier’s work on lambda calculus, continuations, and program transformations

Thank you Olivier