Asymptotic Improvement through Delimited Control
Fast Generic Search with Effect Handlers

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(Joint work with Sam Lindley and John Longley)
The Fundamental Efficiency of Effect Handlers

We consider whether a language with effect handlers admit essential expressiveness differences over a “pure” language.

The question

Let $\mathcal{L}_{\text{eff}}$ a language with effect handlers, and $\mathcal{L} \subset \mathcal{L}_{\text{eff}}$ the fragment modulo effect handlers. Does $\mathcal{L}_{\text{eff}}$ admit asymptotically more efficient programs than $\mathcal{L}$?
We consider whether a language with effect handlers admit essential expressiveness differences over a “pure” language.

The question

Let $L_{\text{eff}}$ a language with effect handlers, and $L \subseteq L_{\text{eff}}$ the fragment modulo effect handlers. Does $L_{\text{eff}}$ admit asymptotically more efficient programs than $L$?

Spoiler alert: the answer is YES. Specifically $O(2^n)$ vs $\Omega(n2^n)$.

To answer positively, it suffices to find one such program. We shall use generic search as our program.
We consider whether a language with effect handlers admit essential expressiveness differences over a “pure” language.

**The question**

Let $\mathcal{L}_{\text{eff}}$ a language with effect handlers, and $\mathcal{L} \subseteq \mathcal{L}_{\text{eff}}$ the fragment modulo effect handlers. Does $\mathcal{L}_{\text{eff}}$ admit asymptotically more efficient programs than $\mathcal{L}$?

**Spoiler alert:** the answer is **YES**. Specifically $\mathcal{O}(2^n)$ vs $\Omega(n2^n)$.

To answer positively, it suffices to find one such program. We shall use *generic search* as our program.

Take $\mathcal{L}$ to be cbv PCF and endow it with effect handlers to obtain $\mathcal{L}_{\text{eff}}$. 

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**The Fundamental Efficiency of Effect Handlers**

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The Generic Search Problem

**Problem:** Given a boolean-valued predicate $P$ on a space $\mathbb{B}^n$ of boolean vectors of length $n$ (for some fixed $n \in \mathbb{N}$), return the number of such vectors $p$ for which $P(p) = \text{true}$. Thus for each $n$, we ask for an implementation of

$$\text{count}_n : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}$$

There is but one rule:

No change of types is allowed! (Longley and Normann 2015)

This rules out tricks such as

- CPS conversion
- Implementing an interpreter for $\mathcal{L}_{\text{eff}}$ in $\mathcal{L}$
A boring constant predicate

\[ tt_0 : (\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool} \]

\[ tt_0 = \lambda p. \text{true} \]

Admits a with no queries model
A slightly more interesting constant predicate

\[ tt_2 : (\text{Nat} \to \text{Bool}) \to \text{Bool} \]
\[ tt_2 \equiv \lambda p. p \, 0; \, p \, 1; \, \text{true} \]

Admits a finite model with no repeated queries
A non-constant predicate

\[ tf_3 : (Nat \to Bool) \to Bool \]
\[ tf_3 \equiv \lambda p. \text{if } p \text{ 1 then if } p \text{ 0 then } p \text{ 2 else false else if } p \text{ 2 then true else } p \text{ 0} \]

Admits a finite model with no repeated queries
Possibly divergent predicate

$$div_0 : (\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}$$

$$div_0 = \text{rec } div_0 p. \text{if } p \ 0 \ \text{then } div_0 p \ \text{else} \ \text{false}$$

Admits an infinite model with repeated queries
Restriction to $n$-standard predicates

We restrict our analysis to predicates whose models are “$n$-standard”; informally

- A perfect binary tree of height $n > 0$, whose interior nodes are queries and leaves are answers.
- Contains every query $?j$ for $j \in \{0, \ldots, n - 1\}$.
- No repeated queries along any path in the model.

For example

$$tt_3 = \lambda p. p\ 0;\ p\ 1;\ p\ 2;\ true$$

is 3-standard because its model is 3-standard
A pure generic search procedure

A possible implementation of generic search in $\mathcal{L}$

\[ \text{count}_n : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) \to \text{Nat} \]

\[ \text{count}_n \triangleq \lambda\text{pred}.\text{count'} n (\lambda i. \perp) \]

where

\[ \text{count'} 0 \quad p \triangleq \text{if pred } p \text{ then } 1 \text{ else } 0 \]

\[ \text{count'} (1 + n) \quad p \triangleq \text{count'} n (\lambda i. \text{if } i = n \text{ then } \text{true} \text{ else } p \ i) + \text{count'} n (\lambda i. \text{if } i = n \text{ then } \text{false} \text{ else } p \ i) \]
A pure generic search procedure

A possible implementation of generic search in $\mathcal{L}$

\[ count_n : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) \to \text{Nat} \]

\[ count_n \overset{\text{def}}{=} \lambda \text{pred}. \, \text{count'} \, n \,(\lambda i. \bot) \]

where

\[ \begin{align*}
\text{count'} \, 0 \; p & \overset{\text{def}}{=} \text{if pred p then 1 else 0} \\
\text{count'} \, (1 + n) \; p & \overset{\text{def}}{=} \text{count'} \, n \,(\lambda i. \text{if } i = n \text{ then true else } p \, i) \\
& \quad + \text{count'} \, n \,(\lambda i. \text{if } i = n \text{ then false else } p \, i)
\end{align*} \]

Example \( \text{count}_3 \, tt_3 \):
A pure generic search procedure

A possible implementation of generic search in $\mathcal{L}$

$count_n : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}$

$count_n \equiv \lambda \text{pred}. count' n \ (\lambda i. \bot)$

where

$count' 0 \ p \equiv \ \text{if} \ \text{pred} \ p \ \text{then} \ 1 \ \text{else} \ 0$

$count' (1 + n) \ p \equiv \ count' n (\lambda i. \text{if} \ i = n \ \text{then} \ \text{true} \ \text{else} \ p \ i)$

$+ \ count' n (\lambda i. \text{if} \ i = n \ \text{then} \ \text{false} \ \text{else} \ p \ i)$

Example $count_3 \ tt_3$: reaches the first leaf
A pure generic search procedure

A possible implementation of generic search in \( \mathcal{L} \)

\[
\text{count}_n : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) \to \text{Nat} \\
\text{count}_n \doteq \lambda \text{pred}. \text{count}' \ n \ (\lambda i. \bot)
\]

where

\[
\text{count}' \ 0 \quad \text{p} \doteq \text{if} \ \text{pred} \ \text{p} \ \text{then} \ 1 \ \text{else} \ 0 \\
\text{count}' \ (1 + n) \ \text{p} \doteq \text{count}' \ n \ (\lambda i. \text{if} \ i = n \ \text{then} \ \text{true} \ \text{else} \ \text{p} \ i) \\
+ \text{count}' \ n \ (\lambda i. \text{if} \ i = n \ \text{then} \ \text{false} \ \text{else} \ \text{p} \ i)
\]

Example \( \text{count}_3 \ \text{tt}_3 \): computation restarts
A pure generic search procedure

A possible implementation of generic search in \( L \)

\[
\text{\( count_n : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) \to \text{Nat} \)}
\]

\[
\text{\( count_n \triangleq \lambda \text{pred}. \text{count'} n (\lambda i. \bot) \)}
\]

where

\[
\text{\( \text{count'} 0 \quad p \triangleq \text{if pred p then 1 else 0} \)}
\]

\[
\text{\( \text{count'} (1 + n) \quad p \triangleq \text{count'} n (\lambda i. \text{if } i = n \text{ then true else p i}) \)}
\]

\[
+ \text{count'} n (\lambda i. \text{if } i = n \text{ then false else p i})
\]

Example \( \text{count}_3 \, tt_3 \): reaches the second leaf
A pure generic search procedure

A possible implementation of generic search in \( \mathcal{L} \)

\[
\text{count}_n : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat} \\
\text{count}_n \doteq \lambda \text{pred}. \text{count}' \ n \ (\lambda i. \bot) \\
\text{where} \\
\text{count}' \ 0 \quad \text{p} \doteq \text{if pred p then 1 else 0} \\
\text{count}' \ (1 + n) \text{ p} \doteq \text{count}' \ n \ (\lambda i. \text{if } i = n \text{ then true else p i}) + \text{count}' \ n \ (\lambda i. \text{if } i = n \text{ then false else p i})
\]

Example \( \text{count}_3 \ tt_3 \): computation restarts

![Diagram](image-url)
A pure generic search procedure

A possible implementation of generic search in $\mathcal{L}$

$\text{count}_n : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) \to \text{Nat}$

$\text{count}_n \equiv \lambda \text{pred} \cdot \text{count}'\ n\ (\lambda i.\bot)$

where

$\text{count}'\ 0\ p \equiv \text{if pred p then 1 else 0}$

$\text{count}'\ (1 + n)\ p \equiv \text{count}'\ n\ (\lambda i.\text{if } i = n \text{ then true else p} i)$

$\quad + \text{count}'\ n\ (\lambda i.\text{if } i = n \text{ then false else p} i)$

Example $\text{count}_3\ tt_3$: reaches the third leaf, etc…
The effectful generic search procedure

For the efficient implementation of generic search in $\mathcal{L}_{eff}$, we require one operation; fix $\Sigma \doteq \{\text{Branch : } \langle \rangle \to \text{Bool}\}$

\[
\begin{align*}
count : ((\text{Nat} \to \text{Bool}) \to \text{Bool}) & \to \text{Nat} \\
count \doteq \lambda \text{pred. handle pred (}\lambda n. \text{do Branch)} \text{ with} \\
& \text{val } x \leftrightarrow \text{if } x \text{ then } 1 \text{ else } 0 \\
& \text{Branch } \langle \rangle \ r \leftrightarrow r \text{ true } + r \text{ false}
\end{align*}
\]
The effectful generic search procedure

For the efficient implementation of generic search in $L_{eff}$, we require one operation; fix $\Sigma \doteq \{\text{Branch : } \langle \rangle \rightarrow \text{Bool}\}$

\[
\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}
\]
\[
\text{count} \doteq \lambda \text{pred}. \text{handle pred} (\lambda n. \text{do Branch}) \text{ with}
\]
\[
\text{val } x \mapsto \text{if } x \text{ then } 1 \text{ else } 0
\]
\[
\text{Branch } \langle \rangle \ r \mapsto r \text{ true } + r \text{ false}
\]

Example count $tt_3$:
The effectful generic search procedure

For the efficient implementation of generic search in $L_{\text{eff}}$, we require one operation; fix $\Sigma \doteq \{ \text{Branch : } \langle \rangle \rightarrow \text{Bool} \}$

\[
\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}
\]
\[
\text{count} \doteq \lambda \text{pred}. \text{handle pred} (\lambda n. \text{do Branch}) \text{ with }
\]
\[
\text{val } x \mapsto \text{if } x \text{ then } 1 \text{ else } 0
\]
\[
\text{Branch } \langle \rangle \text{ r } \mapsto r \text{ true } + r \text{ false}
\]

Example count $tt_3$: reaches the first leaf
The effectful generic search procedure

For the efficient implementation of generic search in $\mathcal{L}_{eff}$, we require one operation; fix $\Sigma \doteq \{\text{Branch : } \langle \rangle \rightarrow \text{Bool}\}$

$$\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}$$

$$\text{count} \doteq \lambda \text{pred}. \text{handle pred} \ (\lambda \text{n}. \text{do} \text{ Branch}) \text{ with }$$

$$\text{val} \ x \ \mapsto \ \text{if} \ x \ \text{then} \ 1 \ \text{else} \ 0$$

Branch $\langle \rangle \ r \ \mapsto \ r \ \text{true} + r \ \text{false}$

Example count $tt_3$: computation backtracks
The effectful generic search procedure

For the efficient implementation of generic search in $L_{eff}$, we require one operation; fix $\Sigma \doteq \{ \text{Branch} : \langle \rangle \rightarrow \text{Bool} \}$

\[
\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}
\]
\[
\text{count} \doteq \lambda \text{pred}. \text{handle pred} (\lambda n. \text{do Branch}) \text{ with } \\
\quad \text{val } x \mapsto \text{if } x \text{ then } 1 \text{ else } 0 \\
\quad \text{Branch } \langle \rangle r \mapsto r \text{ true } + r \text{ false}
\]

Example $\text{count } tt_3$: reaches the second leaf

![Tree diagram showing the reachability of the second leaf for $tt_3$.]
The effectful generic search procedure

For the efficient implementation of generic search in $\mathcal{L}_{\text{eff}}$, we require one operation; fix $\Sigma \doteq \{\text{Branch : } \langle \rangle \rightarrow \text{Bool}\}$

\[
\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat} \\
\text{count} \doteq \lambda \text{pred}. \text{handle pred} (\lambda n. \text{do Branch}) \text{ with} \\
\quad \text{val} \ x \mapsto \text{if } x \text{ then } 1 \text{ else } 0 \\
\quad \text{Branch } \langle \rangle \ r \mapsto r \text{ true } + r \text{ false}
\]

Example count $tt_3$: computation backtracks
The effectful generic search procedure

For the efficient implementation of generic search in $\mathcal{L}_{eff}$, we require one operation; fix $\Sigma = \{\text{Branch} : \langle \rangle \rightarrow \text{Bool}\}$

\[
\text{count} : ((\text{Nat} \rightarrow \text{Bool}) \rightarrow \text{Bool}) \rightarrow \text{Nat}
\]
\[
\text{count} \doteq \lambda \text{pred}. \text{handle pred} (\lambda n. \text{do Branch}) \text{ with val} x \mapsto \text{if } x \text{ then } 1 \text{ else } 0
\]
\[
\text{Branch } \langle \rangle \text{ r } \mapsto \text{r true } + \text{ r false}
\]

Example count $tt_3$: reaches the third leaf, etc...
Main theorem

**Theorem**

1. For every $n$-standard predicate $\text{pred}$, the generic counting procedure has at most time complexity

   $$\text{DTime}(\text{count pred}) = \sum_{bs \in \mathbb{B}^*, |bs| \leq n} \text{steps}(t)(bs) + O(2^n)$$

2. Every generic counting function $\text{count} \in \mathcal{L}$ has for every $n$-standard predicate $\text{pred}$ at least time complexity

   $$\text{DTime}(\text{count pred}) = \sum_{bs \in \mathbb{B}^*, |bs| \leq n} 2^{n-|bs|} \text{steps}(t)(bs) + O(n2^n)$$

Here $t$ denotes the model of pred, and $\text{steps}(t)(bs)$ computes the number of reduction steps used to arrive at the query or answer node determined by $bs$. 
Define suitable machine configuration computing functions

\[
\text{arrive, depart} : \mathbb{B}^* \times \text{Model} \rightarrow \text{Conf}
\]

**Lemma**

Suppose \( t \) is a model of a \( n \)-standard predicate, then for every boolean list \( bs \in \mathbb{B}^* \)

\[
\text{arrive}(bs, t) \rightarrow \sum_{|bs| \leq n} \text{steps}(t)(bs) + 2^n - |bs| \quad \text{depart}(bs, t)
\]

**Proof.**

Proof by downward induction on the list of booleans \( bs \).
Suppose that we have an arbitrary implementation of generic search \( count \in \mathcal{L} \). Pick any \( n \)-standard predicate \( pred \) and look at the computation arising from \( count \ pred \). Now we need to show that

**Lemma (Every leaf is visited (A))**

*The computation \( (count \ pred) \) visits every leaf in the model of \( pred \).*

**Lemma (No shared computation (B))**

*If \( p \) and \( p' \) are distinct points then their subcomputations are disjoint.*

Since each subcomputation has length at least \( \Omega(n) \) the entire computation must have at least length \( \Omega(n2^n) \).
Consider a 3-standard predicate \textit{seven} (has seven true leaves)

Any \(n\)-standard predicate has \(2^n\) threads, and every thread consists of \(n+1\) sections.

Proof of Lemma A.

By contradiction: pick a leaf that has no thread; negate the value at the leaf; tweak the predicate accordingly; observe a wrong result.
Consider a 3-standard predicate *seven* (has seven true leaves)

\[
\text{Thread } \models \{ \text{pred } p \rightsquigarrow^* \mathcal{E}_0[p_0], \\
\mathcal{E}_0[\text{true}] \rightsquigarrow^* \mathcal{E}_1[p_1] \\
\mathcal{E}_1[\text{false}] \rightsquigarrow^* \mathcal{E}_2[p_2], \\
\mathcal{E}_2[\text{true}] \to \text{false} \}
\]

Any \( n \)-standard predicate has \( 2^n \) threads, and every thread consists of \( n + 1 \) sections.
Threads and sections

Consider a 3-standard predicate seven (has seven true leaves)

Thread $\equiv \{ \text{pred } p \leadsto^* \mathcal{E}_0[p 0], \mathcal{E}_0[\text{true}] \leadsto^* \mathcal{E}_1[p 1], \mathcal{E}_1[\text{false}] \leadsto^* \mathcal{E}_2[p 2], \mathcal{E}_2[\text{true}] \rightarrow \text{false} \}$

Any $n$-standard predicate has $2^n$ threads, and every thread consists of $n + 1$ sections.

Proof of Lemma A.

By contradiction: pick a leaf that has no thread; negate the value at the leaf; tweak the predicate accordingly; observe a wrong result.
No shared computation

Every section has a unique successor

Proof.
Follows by definition of section and the semantics being deterministic.

Every section has a single predecessor

Proof.
By direct calculation on the reduction sequence induced by a section.
Summary and future work

In summary

- We have defined two languages $\mathcal{L}$ and $\mathcal{L}_{eff}$
- We have demonstrated that $\mathcal{L}_{eff}$ provides strictly more efficient implementations of generic search than $\mathcal{L}$ ($O(2^n)$ vs $\Omega(n2^n)$)
- …which establish a new complexity result for control operators
- Intuition: control operators build in support for backtracking.

Future considerations

- Perform empirical experiments to observe the result in practice (Daniels 2016)
- Study the robustness of the result, i.e. what feature(s) can we add to $\mathcal{L}$ whilst retaining an efficiency gap between $\mathcal{L}$ and $\mathcal{L}_{eff}$?
- Generalise the result to all conceivable effective models of computations